

Introducción al Álgebra (MA 1101)

Control 4 Pauta Problems 1

a) Calcular $\sum_{k=0}^{2m} \sum_{j=0}^{4m-2k} (-1)^k \binom{2m}{2m-k} \binom{4m-2k}{j}$ \rightarrow no depende de j

$$= \sum_{k=0}^{2m} (-1)^k \binom{2m}{2m-k} \underbrace{\sum_{j=0}^{4m-2k} \binom{4m-2k}{j}}_{\text{Binomio } (1+1)^{4m-2k}} 1^j \quad \text{y } \binom{2m}{2m-k} = \binom{2m}{k} \text{ Propiedad}$$

(1.0) \rightarrow

$$= \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} 2^{4m-2k} = \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} (2^2)^{2m-k} = (2^2 - 1)^{2m} = 3^{2m}$$

(2.0) \rightarrow Binomio

b) Calcule $\sum_{k=1}^m \sum_{j=1}^{k^2} \frac{(m+k^2)}{(m+j-1)(m+j)}$ \rightarrow independiente de j

$$= \sum_{k=1}^m \left[(m+k^2) \sum_{j=1}^{k^2} \frac{1}{(m+j-1)(m+j)} \right] = \sum_{k=1}^m (m+k^2) \sum_{j=1}^{k^2} \left(\frac{1}{m+j-1} - \frac{1}{m+j} \right)$$

(1.0) \rightarrow Telescópica

$$= \sum_{k=1}^m (m+k^2) \left(\frac{1}{m} - \frac{1}{m+k^2} \right) = \sum_{k=1}^m (m+k^2) \frac{k^2}{m(m+k^2)}$$

(2.0) \rightarrow

$$= \frac{1}{m} \sum_{k=1}^m k^2 = \frac{1}{m} \frac{m(m+1)(2m+1)}{6} = \frac{(m+1)(2m+1)}{6}$$

Suma conocida.

Punto Problema 2

a) Demostrar, sin usar inducción, que

$$\sum_{k=0}^n (-1)^k \frac{\binom{n}{k}}{(k+1)(k+2)} = \frac{1}{n+2}, \quad \forall n \in \mathbb{N}$$

En efecto $\sum_{k=0}^n (-1)^k \frac{n!}{k!(n-k)!(k+1)(k+2)} = \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!(k+2)!} =$

Completando el número combinatorio $= \frac{1}{(n+1)(n+2)} \sum_{k=0}^n (-1)^k \frac{n!(n+1)(n+2)}{(k+2)!(n-k)!}$

(1.5) $\rightarrow = \frac{1}{(n+1)(n+2)} \sum_{k=0}^n (-1)^k \frac{(n+2)!}{(k+2)!(n-k)!} = \frac{1}{(n+1)(n+2)} \sum_{k=0}^n (-1)^k \binom{n+2}{k+2}$

(0.5) \rightarrow cambiando índices $= \frac{1}{(n+1)(n+2)} \sum_{k=2}^{n+2} (-1)^k \binom{n+2}{k}$ donde $(-1)^k = (-1)^{k-2} = (-1)^k$

(1.0) $\rightarrow = \frac{1}{(n+1)(n+2)} \sum_{k=2}^{n+2} (-1)^k \binom{n+2}{k} = \frac{1}{(n+1)(n+2)} \left[\sum_{k=0}^{n+2} (-1)^k \binom{n+2}{k} - \binom{n+2}{0} + \binom{n+2}{1} \right]$
 Binomio $(-1)^k = 0$
 $= \frac{1}{(n+1)(n+2)} \left[0 - \underbrace{\binom{n+2}{0}}_1 + \underbrace{\binom{n+2}{1}}_{n+2} \right] = \frac{1}{(n+1)(n+2)} \left[-1 + n+2 \right]$

(1.0) \rightarrow Así $\sum_{k=0}^n (-1)^k \frac{\binom{n}{k}}{(k+1)(k+2)} = \frac{1}{n+2}$

b) $M = \{A \subseteq \mathbb{Q} / |A| = 2\}$

Consideremos $A \in M$ en que $A = \{p, q\}$ con $p < q$ y $|A| = 2$

(0.5) \rightarrow Se puede definir $f: M \rightarrow \mathbb{Q} \times \mathbb{Q}$
 $A = \{p, q\} \rightarrow f(\{p, q\}) = (p, q)$ numerable

Claramente f es inyectiva y entonces $|M| \leq |\mathbb{Q} \times \mathbb{Q}| = |\mathbb{N}|$

(1.5) Además M es infinito (por ejemplo los pares $(0, q)$ con $q \in \mathbb{Q}^+$ son infinitos), entonces $|\mathbb{N}| < |M| \leq$ concluyendo que $|M| = |\mathbb{N}|$ por lo tanto M es numerable.